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Properties of Ergodic Projection for Quantum Dynamical Semigroups

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We study properties of ergodic projection for quantum dynamical semigroups on W^*-algebras. In particular, we describe the normal and singular parts of this projection, characterize normal invariant functionals, and derive some conclusions for ergodic semigroups.

INTRODUCTION

We continue the analysis of ergodic projection for a semigroup of normal positive contractions on a W^*-algebra begun in Luczak (1995). The normal and singular parts of such a projection are shown to be projections, too, into the fixed-point space of the semigroup, and the set of normal invariant functionals is characterized by means of the normal part of the ergodic projection. Moreover, it is proved that this normal part is unique, and that it coincides with the whole ergodic projection on some W^*-subalgebra of M. Finally, some conclusions are drawn for ergodic quantum dynamical systems. It seems to be worth noting that the approach adopted in this paper allows us to disregard any continuity properties of the action of the semigroup on the algebra.

1. PRELIMINARIES AND NOTATION

Throughout the paper, M will stand for a W^*-algebra with predual M_* and identity $\mathbf{1}$. Let $\mathbb{G}$ be a semigroup and let α be a representation of $\mathbb{G}$ into the set of linear bounded mappings on M. We shall assume that the α_g are

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normal positive contractions. The semigroup (α_g: $g \in \mathbb{G}$) is then called a quantum dynamical semigroup. A bounded linear functional φ on M is said to be $\mathbb{G}$-invariant if $\varphi \circ \alpha_g = \varphi$ for all $g \in \mathbb{G}$. We write $M^{*\mathbb{G}}$ (resp. $M_*^{\mathbb{G}}$) for the set of all $\mathbb{G}$-invariant (resp. normal $\mathbb{G}$-invariant) bounded linear functionals on M, and $M^{\mathbb{G}}$ stands for the space of all $\mathbb{G}$-invariant elements of M, i.e., $M^{\mathbb{G}} = \{x \in M: \alpha_g(x) = x, g \in \mathbb{G}\}$.

A bounded linear mapping ε on M is called an ergodic projection if:

(i) $\varepsilon(M) \subset M^{\mathbb{G}}$, $\varepsilon^2 = \varepsilon$
(ii) $\varepsilon \circ \alpha_g = \alpha_g \circ \varepsilon$ for all $g \in \mathbb{G}$;
(iii) $\varepsilon \in \overline{\text{conv}}\ \{\alpha_g: g \in \mathbb{G}\}$

where the closure of the convex hull is taken in the point-σ-weak topology on the space $\mathbb{B}(M)$ of all bounded linear mappings on M, i.e., the topology given by the system of seminorms $\{\|\cdot\|_{\varphi,x}: \varphi \in M_*, x \in M\}$

$$\|\Phi\|_{\varphi,x} := |\varphi(\Phi x)|, \qquad \Phi \in \mathbb{B}(M)$$

It follows that ε is positive and $\varepsilon(M) = M^{\mathbb{G}}$.

Remark 1. If ε is an ergodic projection, then:
(a) For each $\varphi \in M_*^{\mathbb{G}}$, $\varphi \circ \varepsilon = \varphi$.
(b) For each $\varphi \in M^*$, $\varphi \circ \varepsilon \in M^{*\mathbb{G}}$.
Indeed, (a) is a consequence of (iii), and (b) follows immediately from (ii).

Remark 2. It can easily be shown that if an ergodic projection is normal, then it is unique.

The problem of the existence of an ergodic projection for semigroups of mappings on a W^*-algebra was considered on various levels of generality in (Frigerio, 1978; Frigerio and Verri, 1982; Kümmerer and Nagel, 1979; Thomsen, 1985; Watanabe, 1979). In this paper, taking existence for granted, we investigate its structure and properties.

2. THE NORMAL AND SINGULAR PARTS OF ERGODIC PROJECTION

Let

$$\varepsilon = \varepsilon_n + \varepsilon_s \tag{1}$$

be the decomposition of ergodic projection ε into its normal and singular parts, respectively (Takesaki, 1979, p. 128). We have:

Theorem 1. (i) ε_n and ε_s are (α_g)-invariant projections from M into $M^{\mathbb{G}}$ such that $\varepsilon_n \circ \varepsilon_s = \varepsilon_g \circ \varepsilon_n = 0$.

(ii) $M_*^{\mathbb{G}} = \{\varphi \circ \varepsilon_n : \varphi \in M^* \}$.

Proof. (i) In Luczak (1995, Theorem 5) it is shown that

$$\alpha_g \circ \varepsilon_n = \varepsilon_n, \qquad \alpha_g \circ \varepsilon_s = \varepsilon_s$$

and thus ε_n and ε_s map M into $M^{\mathbb{G}}$; hence

$$\varepsilon \circ \varepsilon_n = \varepsilon_n, \qquad \varepsilon \circ \varepsilon_s = \varepsilon_s \tag{2}$$

Furthermore,

$$\varepsilon_n + \varepsilon_s = \varepsilon = \varepsilon^2 = (\varepsilon_n + \varepsilon_s)(\varepsilon_n + \varepsilon_s) = \varepsilon_n^2 + \varepsilon_n \circ \varepsilon_s + \varepsilon_s \circ \varepsilon_n + \varepsilon_s^2$$

Multiplying both sides of (1) by ε_n on the right, we obtain, on account of (2),

$$\varepsilon_n = \varepsilon_n^2 + \varepsilon_s \circ \varepsilon_n \tag{3}$$

which shows that $\varepsilon_s \circ \varepsilon_n$ is a normal map.

Analogously, multiplying both sides of (1) by ε_s on the right, we obtain

$$\varepsilon_s = \varepsilon_n \circ \varepsilon_s + \varepsilon_s^2 \tag{4}$$

and since $\varepsilon_n \circ \varepsilon_s$ is clearly singular, we get that ε_s^2 is singular, too.

The contractivity of α_g yields

$$\alpha_g(\mathbf{1}) \leq \mathbf{1}$$

i.e.

$$\mathbf{1} - \alpha_g(\mathbf{1}) \geq 0.$$

As ε is (α_g)-invariant, we have

$$\varepsilon(\mathbf{1} - \alpha_g(\mathbf{1})) = 0$$

Since ε_n and ε_s are positive, it follows that

$$\varepsilon_n(\mathbf{1} - \alpha_g(\mathbf{1})) = \varepsilon_s(\mathbf{1} - \alpha_g(\mathbf{1})) = 0$$

in other words

$$\varepsilon_n(\mathbf{1}) = \varepsilon_n \circ \alpha_g(\mathbf{1}), \qquad \varepsilon_s(\mathbf{1}) = \varepsilon_s \circ \alpha_g(\mathbf{1}) \tag{5}$$

We have

$$\varepsilon_n + \varepsilon_s = \varepsilon_n \circ \alpha_g + \varepsilon_s \circ \alpha_g = \varepsilon_n \circ \alpha_g + \Phi_n^{(g)} + \Phi_s^{(g)} \tag{6}$$

where

$$\varepsilon_s \circ \alpha_g = \Phi_n^{(g)} + \Phi_s^{(g)} \tag{7}$$

is the decomposition of $\varepsilon_s \circ \alpha_g$ into its normal and singular parts. From (6) we obtain

$$\varepsilon_n = \varepsilon_n \circ \alpha_g + \Phi_n^{(g)}, \qquad \varepsilon_s = \Phi_s^{(g)} \tag{8}$$

and (5) yields

$$\Phi_n^{(g)}(\mathbf{1}) = 0$$

Since $\phi_n^{(g)}$ is positive (as the normal part of positive map $\varepsilon_s \circ \alpha_g$), the last equality shows that

$$\Phi_n^{(g)} = 0$$

and now (7) and (8) yield the equalities

$$\varepsilon_n = \varepsilon_n \circ \alpha_g, \qquad \varepsilon_s = \varepsilon_s \circ \alpha_g \tag{9}$$

From the first equality above and property (iii) in the definition of ergodic projection we obtain

$$\varepsilon_n = \varepsilon_n \circ \varepsilon$$

which gives

$$\varepsilon_n = \varepsilon_n \circ (\varepsilon_n + \varepsilon_s) = \varepsilon_n^2 + \varepsilon_n \circ \varepsilon_s \tag{10}$$

As we have noticed before, $\varepsilon_n \circ \varepsilon_s$ is singular, ε_n and ε_n^2 are normal, so

$$\varepsilon_n \circ \varepsilon_s = 0$$

and, consequently, by (10), (4), and (3)

$$\varepsilon_n = \varepsilon_n^2, \qquad \varepsilon_s = \varepsilon_s^2, \qquad \varepsilon_s \circ \varepsilon_n = 0$$

showing (i).

(ii) For each $\varphi \in M_*$ we have on account of (9)

$$\varphi \circ \varepsilon_n \circ \alpha_g = \varphi \circ \varepsilon_n$$

so $\varphi \circ \varepsilon_n \in M_*^{\mathbb{G}}$.

Conversely, for $\varphi \in M_*^{\mathbb{G}}$.

$$\varphi = \varphi \circ \varepsilon = \varphi \circ \varepsilon_n + \varphi \circ \varepsilon_s,$$

and since $\varphi, \varphi \circ \varepsilon_n$ are normal and $\varphi \circ \varepsilon_s$ is singular, we get $\varphi \circ \varepsilon_s = 0$; thus

$$\varphi = \varphi \circ \varepsilon_n$$

which finishes the proof of (ii). ■

The above theorem has a number of corollaries.

Corollary 1. There is a nonzero normal $\mathbb{G}$-invariant linear functional if and only if $\varepsilon_n \neq 0$.

This follows at once from part (ii) of the Theorem. ■

Let us recall that the recurrent projection p_r is defined as

$$p_r = \sup\{s(\varphi): \varphi \in M_*^{\mathbb{G}}, \varphi \geq 0\}$$

where $s(\varphi)$ stands for the support of a normal positive linear functional φ (Evans and Høegh-Krohn, 1978; Frigerio and Verri, 1982; Groh, 1986; Łuczak, 1995). Letting $s(\varepsilon_n)$ denote the support of the normal positive map ε_n (i.e., the smallest projection e such that $\varepsilon_n(e) = \varepsilon_n(\mathbf{1})$, we get:

Corollary 2. $s(\varepsilon_n) = p_r$.

Corollary 3. $M^{\mathbb{G}} = \varepsilon_n(M) \oplus \varepsilon_s(M)$. Moreover, if (α_g) is ergodic (i.e., dim $M^{\mathbb{G}} = 1$), then ε is either normal or singular.

Indeed, let $x \in \varepsilon_n(M) \cap \varepsilon_n(M)$. Then

$$x = \varepsilon_n(x) \qquad \text{and} \qquad x = \varepsilon_s(x)$$

Thus

$$x = \varepsilon_n(x) = \varepsilon_n(\varepsilon_s(x)) = 0$$

and since $M^{\mathbb{G}} = \varepsilon(M)$, the decomposition follows. The second part of the corollary follows from the first. ■

Corollary 4. dim $M_*^{\mathbb{G}} = 1$ if and only if dim $\varepsilon_n(M) = 1$.

Assume first that $M_*^{\mathbb{G}} = \{\gamma\omega : \gamma \in \mathbb{C}\}$ for some $\mathbb{G}$-invariant $\omega \neq 0$. For each $\varphi \in M_*$, we have

$$\varphi \circ \varepsilon_n = \gamma(\varphi)\omega$$

which gives

$$\gamma(\varphi) = \frac{\varphi \circ \varepsilon_n(x_0)}{\omega(x_0)}$$

for some x_0 such that $\omega(x_0) \neq 0$. Accordingly, for $x \in M$

$$\varphi(\varepsilon_n(x)) = \frac{\varphi(\varepsilon_n(x_0))}{\omega(x_0)}\,\omega(x) = \varphi\left(\omega(x)\varepsilon_n\left(\frac{x_0}{\omega(x_0)}\right)\right)$$

so that

$$\varepsilon_n(x) = \omega(x)z_0 \qquad \text{with} \qquad z_0 = \varepsilon_n\left(\frac{x_0}{\omega(x_0)}\right)$$

showing that $\varepsilon_n(M)$ is one-dimensional.

Conversely, if dim $\varepsilon_n(M) = 1$, then for some $z_0 \neq 0$

$$\varepsilon_n(x) = \omega(x)z_0, \qquad x \in M$$

and so, for $\varphi \in M_*$

$$\varphi \circ \varepsilon_n = \varphi(z_0)\omega$$

which shows that $M_*^{\mathbb{G}}$ is one-dimensional. ■

Our next aim is to show that the normal part is the same for every ergodic projection. Namely, we shall prove:

Theorem 2. Let $\varepsilon = \varepsilon_n + \varepsilon_s$ and $\varepsilon' = \varepsilon_n' + \varepsilon_s'$ be two ergodic projections. Then $\varepsilon_n = \varepsilon_n'$

Proof. For each $\varphi \in M_*$, $\varphi \circ \varepsilon_n' \in M_*^{\mathbb{G}}$ and thus we have

$$\varphi \circ \varepsilon_n' \circ \varepsilon_n = \varphi \circ \varepsilon_n'$$

which yields

$$\varepsilon_n' \circ \varepsilon_n = \varepsilon_n'$$

By the same token, we obtain

$$\varepsilon_n \circ \varepsilon_n' = \varepsilon_n$$

Since ε and ε_n' are projections into $M_*^{\mathbb{G}}$, we have

$$\varepsilon_n \circ \varepsilon_n' = \varepsilon_n', \qquad \varepsilon_n' \circ \varepsilon_n = \varepsilon_n$$

hence

$$\varepsilon_n' = (\varepsilon_n + \varepsilon_s) \circ \varepsilon_n' = \varepsilon_n \circ \varepsilon_n' + \varepsilon_s \circ \varepsilon_n' = \varepsilon_n + \varepsilon_s \circ \varepsilon_n'$$

and

$$\varepsilon_n = (\varepsilon_n' + \varepsilon_s') \circ \varepsilon_n = \varepsilon_n' \circ \varepsilon_n + \varepsilon_s' \circ \varepsilon_n = \varepsilon_n' + \varepsilon_s' \circ \varepsilon_n$$

Consequently,

$$\varepsilon_n' - \varepsilon_n = \varepsilon_s \circ \varepsilon_n' \geq 0$$

and

$$\varepsilon_n - \varepsilon_n' = \varepsilon_s' \circ \varepsilon_n \geq 0$$

so

$$\varepsilon_n = \varepsilon_n'$$

follows. ■

Corollary 5. Let ε, ε', ε_n, ε'_n, ε_s, ε'_s be as before. Then $\varepsilon_s(M) = \varepsilon'_s(M)$ and $\varepsilon_s(x) = \varepsilon'_s(x)$ for $x \in M^{\mathbb{G}}$.

It follows from the decomposition given in Corollary 3 and the equality $\varepsilon_n = \varepsilon'_n$. ■

In the last part of the paper, we assume that the α_g are unital, i.e.,

$$\alpha_g(\mathbf{1}) = \mathbf{1}$$

Theorem 3. Let p_r be the recurrent projection. Then

$$\varepsilon|p_r M p_r = \varepsilon_n|p_r M p_r$$

Proof. We have

$$\mathbf{1} = \varepsilon(\mathbf{1}) = \varepsilon_n(\mathbf{1}) + \varepsilon_s(\mathbf{1})$$

In the proof of Theorem 5 in Łuczak (1995) it was shown that

$$p_r \varepsilon_s(\mathbf{1}) p_r = 0$$

implying that

$$p_r = p_r \varepsilon_n(\mathbf{1}) p_r$$

which, in turn, gives

$$\varepsilon_n(\mathbf{1}) \geq p_r$$

By virtue of Corollary 2, we have

$$\varepsilon_n(p_r) = \varepsilon_n(\mathbf{1}) \geq p_r$$

and by (2)

$$\varepsilon_n(p_r) = \varepsilon(\varepsilon_n(p_r)) \geq \varepsilon(p_r) \geq \varepsilon_n(p_r)$$

which gives the equality

$$\varepsilon(p_r) = \varepsilon_n(p_r)$$

showing that

$$\varepsilon_s(p_r) = 0$$

Since ε_s is positive and p_r is the identity of the W^*-algebra $p_r\ M\ p_r$, it follows that

$$\varepsilon_s|p_r M p_r = 0$$

and, consequently,

$$\varepsilon|p_r M p_r = \varepsilon_n|p_r M p_r \quad \blacksquare$$

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